

On the domination polynomials of cactus chains

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ABSTRACT

Let G be a simple graph of order n . The domination polynomial of G is the polynomial $D(G, x) = \sum_{i=\gamma(G)}^n d(G, i)x^i$, where $d(G, i)$ is the number of dominating sets of G of size i and $\gamma(G)$ is the domination number of G . In this paper we consider cactus chains with triangular and square blocks and study their domination polynomials.

Mathematics Subject Classification: 05C60, 05C69.

Keywords: Domination polynomial; dominating sets; cactus.

1 Introduction

Let $G = (V, E)$ be a simple graph. For any vertex $v \in V(G)$, the *open neighborhood* of v is the set $N(v) = \{u \in V(G) | \{u, v\} \in E(G)\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V(G)$, the open neighborhood of S is $N(S) = \bigcup_{v \in S} N(v)$ and the closed neighborhood of S is $N[S] = N(S) \cup S$. A set $S \subseteq V(G)$ is a *dominating set* if $N[S] = V$ or equivalently, every vertex in $V(G) \setminus S$ is adjacent to at least one vertex in S . The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set in G . For a detailed treatment of these parameters, the reader is referred to [9]. Let $\mathcal{D}(G, i)$ be the family of dominating sets of a graph G with cardinality i and let $d(G, i) = |\mathcal{D}(G, i)|$. The *domination polynomial* $D(G, x)$ of G is defined as $D(G, x) = \sum_{i=\gamma(G)}^{|V(G)|} d(G, i)x^i$, where $\gamma(G)$ is the domination number of G (see [1, 4]). Obviously, the number of dominating sets of a graph G is $D(G, 1)$ (see [3, 12]). Recently the number of the dominating sets of graph G , i.e., $D(G, 1)$ has been considered and studied in [17] with a different approach.

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Domination theory have many applications in sciences and technology (see [9]). Recently the dominating set has found application in the assignment of structural domains in complex protein structures, which is an important task in bio-informatics ([7]).

We recall that the Hosoya index $Z(G)$ of a molecule graph G , is the number of matching sets, and the Merrifield-Simmons index $i(G)$ of graph G , is the number of independent sets. The Hosoya index of a graph has application to correlations with boiling points, entropies, calculated bond orders, as well as for coding of chemical structures. The Merrifield-Simmons index is one of the most popular topological indices in chemistry. For more information of these two indices see [14, 15, 18]. Note that $Z(G)$ and $i(G)$ can be study by the value of matching polynomial and independence polynomial at 1.

In this paper we consider a class of simple linear polymers called cactus chains. Cactus graphs were first known as Husimi trees; they appeared in the scientific literature some sixty years ago in papers by Husimi and Riddell concerned with cluster integrals in the theory of condensation in statistical mechanics [8, 10, 16]. We refer the reader to papers [6, 13] for some aspects of domination in cactus graphs.

A cactus graph is a connected graph in which no edge lies in more than one cycle. Consequently, each block of a cactus graph is either an edge or a cycle. If all blocks of a cactus G are cycles of the same size i , the cactus is i -uniform. A triangular cactus is a graph whose blocks are triangles, i.e., a 3-uniform cactus. A vertex shared by two or more triangles is called a cut-vertex. If each triangle of a triangular cactus G has at most two cut-vertices, and each cut-vertex is shared by exactly two triangles, we say that G is a chain triangular cactus. By replacing triangles in this definitions by cycles of length 4 we obtain cacti whose every block is C_4 . We call such cacti square cacti. Note that the internal squares may differ in the way they connect to their neighbors. If their cut-vertices are adjacent, we say that such a square is an ortho-square; if the cut-vertices are not adjacent, we call the square a para-square.

In Section 2 we study the domination polynomial of the chain triangular cactus with two approach. In Section 3 we study the domination polynomials of chains of squares.

2 Domination polynomials of the chain triangular cactus

We call the number of triangles in G , the length of the chain. An example of a chain triangular cactus is shown in Figure 1. Obviously, all chain triangular cacti of the same length are isomorphic. Hence, we denote the chain triangular cactus of length n by T_n . In this paper we investigate the domination polynomial of T_n by two different approach.

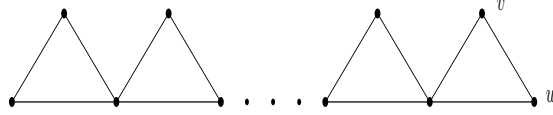


Figure 1: The chain triangular cactus.

2.1 Computation of $D(T_n, x)$ using recurrence relation

In the first subsection, we use results and recurrence relations of the domination polynomial of a graph to find a recurrence relation for $D(T_n, x)$.

We need the following theorem:

Theorem 1.[4] *If a graph G consists of k components G_1, \dots, G_k , then $D(G, x) = \prod_{i=1}^k D(G_i, x)$.*

The vertex contraction G/u of a graph G by a vertex u is the operation under which all vertices in $N(u)$ are joined to each other and then u is deleted (see[19]).

The following theorem is useful for finding the recurrence relations for the domination polynomials of arbitrary graphs.

Theorem 2.[2, 11] *Let G be a graph. For any vertex u in G we have*

$$D(G, x) = xD(G/u, x) + D(G - u, x) + xD(G - N[u], x) - (1 + x)p_u(G, x),$$

where $p_u(G, x)$ is the polynomial counting the dominating sets of $G - u$ which do not contain any vertex of $N(u)$ in G .

Domination polynomial satisfies a recurrence relation for arbitrary graphs which is based on the edge and vertex elimination operations. The recurrence uses composite operations, e.g. $G - e/u$, which stands for $(G - e)/u$.

Theorem 3.[11] *Let G be a graph. For every edge $e = \{u, v\} \in E$,*

$$\begin{aligned} D(G, x) &= D(G - e, x) + \frac{x}{x-1} \left[D(G - e/u, x) + D(G - e/v, x) \right. \\ &\quad - D(G/u, x) - D(G/v, x) - D(G - N[u], x) - D(G - N[v], x) \\ &\quad \left. + D(G - e - N[u], x) + D(G - e - N[v], x) \right]. \end{aligned}$$

We use for graphs $G = (V, E)$ the following vertex operation, which is commonly found in the literature. Let $v \in V$ be a vertex of G . A vertex appending $G + e$ (or $G + \{v, \cdot\}$) denotes the graph $(V \cup \{v'\}, E \cup \{v, v'\})$ obtained from G by adding a new vertex v' and an edge $\{v, v'\}$ to G .

The following theorem gives recurrence relation for the domination polynomial of T_n .

Theorem 4. *For every $n \geq 3$,*

$$D(T_n, x) = (x^2 + 2x)D(T_{n-1}, x) + (x^2 + x)D(T_{n-2}, x),$$

with initial condition $D(T_1, x) = x^3 + 3x^2 + 3x$ and $D(T_2, x) = x^5 + 5x^4 + 10x^3 + 8x^2 + x$.

Proof. Consider the graph T_n as shown in the following Figure 1. Since T_n/u is isomorphic to $T_n - u$ and $p_u(T_n, x) = 0$, by Theorem 2 we have:

$$\begin{aligned} D(T_n, x) &= xD(T_n/u, x) + D(T_n - u, x) + xD(T_n - N[u], x) - (1+x)p_u(T_n, x) \\ &= (x+1)D(T_n/u, x) + xD(T_n - N[u], x) \\ &= (x+1)D(T_{n-1} + e, x) + xD(T_{n-2} + e, x). \end{aligned} \tag{1}$$

Note we use Theorems 1 and 2 to obtain the domination polynomial of the graph $T_{n-1} + e$ (see Figure 2). Suppose that v' be a vertex of degree 1 in graph $T_{n-1} + e$ and let u be its neighbor. Note that in this case $p_u(T_{n-1} + e, x) = 0$. We deduce that for each $n \in \mathbb{N}$, $D(T_{n-1} + e, x) =$

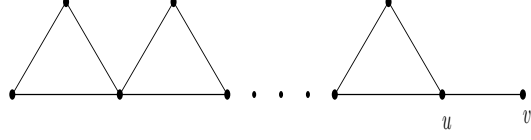


Figure 2: The Graph $T_{n-1} + e$.

$x[D(T_{n-1}, x) + D(T_{n-2} + e, x) + D(T_{n-3} + e, x)]$. Therefore by equation (1) and this equality we have

$$D(T_n, x) = (x^2 + x)(D(T_{n-1}, x) + D(T_{n-3} + e, x)) + (x^2 + 2x)D(T_{n-2} + e, x).$$

Now it's suffices to prove the following equality:

$$(x^2 + x)D(T_{n-3} + e, x) + (x^2 + 2x)D(T_{n-2} + e, x) = xD(T_{n-1}, x) + (x^2 + x)D(T_{n-2}, x).$$

For this purpose we use Theorem 2 for $D(T_{n-1}, x)$. We have

$$xD(T_{n-1}, x) = (x^2 + x)D(T_{n-2} + e, x) + x^2D(T_{n-3} + e, x).$$

Now we use Theorem 2 for v' to obtain domination polynomial of $T_{n-2} + e$, then we have $D(T_{n-2} + e, x) = (1 + x)D(T_{n-2}, x) + xD(T_{n-3} + e, x) - (1 + x)D(T_{n-3} + e, x)$. Therefore the result follows. \square

2.2 Computation of $D(T_n, x)$ by counting the number of dominating sets

In this section we shall obtain a recurrence relation for the domination polynomial of T_n . For this purpose we count the number of dominating sets of T_n with cardinality k . In other words, we first find a two variables recursive formula for $d(T_n, k)$.

Recently by private communication, we found that the following result also appear in [5] but were proved independently.

Theorem 5. *The number of dominating sets of T_n with cardinality k is given by*

$$d(T_n, k) = 2d(T_{n-1}, k - 1) + d(T_{n-1}, k - 2) + d(T_{n-2}, k - 1) + d(T_{n-2}, k - 2).$$

Proof. We shall make a dominating set of T_n with cardinality k which we denote it by \mathcal{T}_n^k . We consider all cases:

Case 1. If \mathcal{T}_n^k contains both of v and w , then we have $\mathcal{T}_n^k = \mathcal{T}_{n-1}^{k-2} \cup \{v, w\}$. In this case we have $d(T_n, k) = d(T_{n-1}, k-2)$.

Case 2. If \mathcal{T}_n^k contains only v or w (say v), then we have $\mathcal{T}_n^k = \mathcal{T}_{n-1}^{k-1} \cup \{v\}$. In this case we have $d(T_n, k) = 2d(T_{n-1}, k-1)$.

Case 3. If \mathcal{T}_n^k contains none of v and w , then we can construct \mathcal{T}_n^k by \mathcal{T}_{n-2}^{k-1} or \mathcal{T}_{n-2}^{k-2} as shown in Figure 3. In this case we have $d(T_n, k) = d(T_{n-2}, k-1) + d(T_{n-2}, k-2)$. By adding all contributions we obtain the recurrence for $d(T_n, k)$. \square

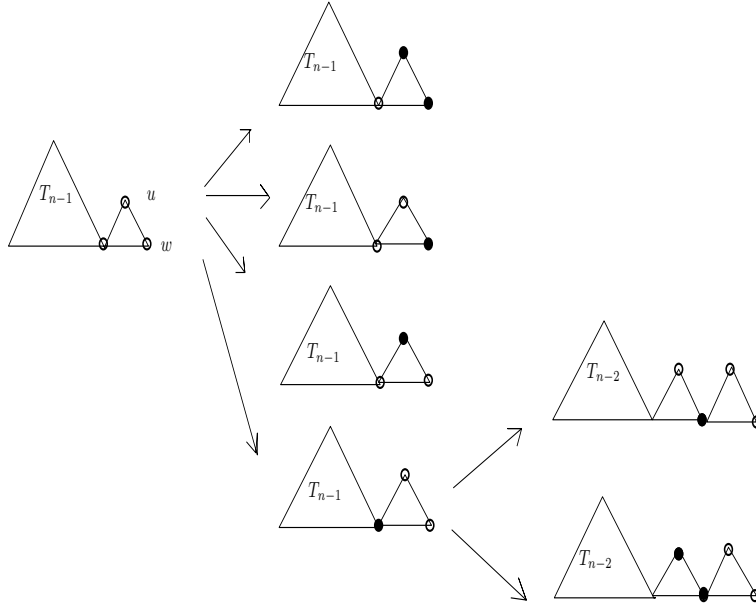


Figure 3: Recurrence relation for $d(T_n, k)$.

Corollary 1. For every $n \geq 3$,

$$D(T_n, x) = (x^2 + 2x)D(T_{n-1}, x) + (x^2 + x)D(T_{n-2}, x).$$

Proof. It follows from Theorem 5 and the definition of the domination polynomial. \square

We mention here the Hosoya index of a graph G is the total number of matchings of G and the Merrifield-Simmons index is the total number of its independent sets. Motivation by these indices, we are interested to count the total number of dominating set of a graph which is equal to $D(G, 1)$. Here we present a recurrence relation to the total number of the chain triangular cactus.

Theorem 6. *The enumerating sequence $\{t_n\}$ for the number of dominating sets in T_n ($n \geq 2$) is*

$$t_n = 3t_{n-1} + 2t_{n-2}$$

with initial values $t_0 = 2$, $t_1 = 7$.

Proof. Since $t_n = D(T_n, 1)$, it follows from Corollary 1. \square

3 Domination polynomials of chains of squares

By replacing triangles in the definitions of triangular cactus, by cycles of length 4 we obtain cacti whose every block is C_4 . We call such cacti, square cacti. An example of a square cactus chain is shown in Figure 4. We see that the internal squares may differ in the way they connect to their neighbors. If their cut-vertices are adjacent, we say that such a square is an ortho-square; if the cut-vertices are not adjacent, we call the square a para-square.

3.1 Domination polynomial of para-chain square cactus graphs

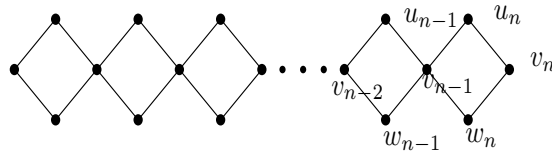


Figure 4: Para-chain square cactus graphs.

In this subsection we consider a para-chain of length n , Q_n , as shown in Figure 4. We shall obtain a recurrence relation for the domination polynomial of Q_n . As usual we denote the

number of dominating sets of Q_n by $d(Q_n, k)$. The following theorem gives a recurrence relation for $D(Q_n, x)$.

We need the following Lemma for finding domination polynomial of the Q_n .

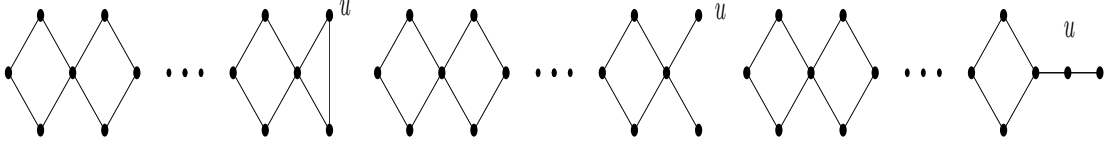


Figure 5: Graphs Q_n^Δ , Q'_n and $Q_n(2)$, respectively.

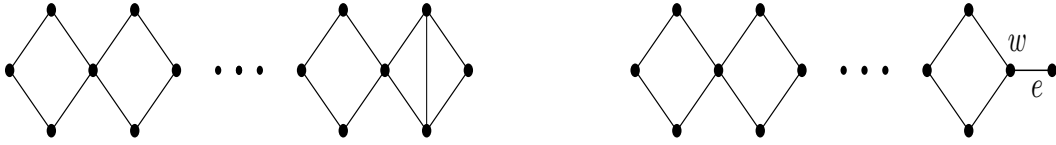


Figure 6: Graphs $(Q_n + e)/w$ and $Q_n + e$, respectively.

Lemma 1. For graphs in figures 5 and 6 have:

- (i) $D(Q_n^\Delta, x) = (1 + x)D(Q_n + e, x) + xD(Q'_{n-1}, x)$, where $D(Q_0^\Delta, x) = x^3 + 3x^2 + 3x$.
- (ii) $D(Q_n(2), x) = x(D(Q_n + e, x) + D(Q_n, x) + D(Q'_{n-1}, x))$, where $D(Q_0(2), x) = x^3 + 3x^2 + x$.
- (iii) $D(Q'_n, x) = (1 + x)D(Q_n + e, x) - xD(Q'_{n-1}, x)$, where $D(Q'_0, x) = x^3 + 3x^2 + x$.
- (iv) $D(Q_n + e, x) = x(D(Q_n, x) + D(Q_{n-1}, x)) + xD(Q'_{n-1}, x) + 2x^2D(Q'_{n-2}, x)$, where $D(Q_1 + e, x) = x^5 + 5x^4 + 9x^3 + 4x^2$.

Proof. The proof of parts (i) and (ii) follow from Theorems 1 and 2 for vertex u in graphs Q_n^Δ and $Q_n(2)$, respectively. Note that in these cases $p_u(G, x) = 0$.

(iii) We use Theorems 1 and 2 for vertex u to obtain domination polynomial of Q'_n , then we have

$$\begin{aligned}
 D(Q'_n, x) &= (1 + x)D(Q_n + e, x) + x^2D(Q'_{n-1}, x) - (1 + x)xD(Q'_{n-1}, x) \\
 &= (1 + x)D(Q_n + e, x) - x^2D(Q'_{n-1}, x).
 \end{aligned}$$

(iv) We use Theorems 1 and 2 for vertex w to obtain domination polynomial of $Q_n + e$, as shown in figure 6 then we have $D(Q_n + e, x) = xD((Q_n + e)/w, x) + xD(Q'_{n-1}, x) + xD(Q_{n-1}, x)$. Now consider the graph $(Q_n + e)/w$ as shown in figure 6. We use Theorems 1 and 3 for $e = \{u, v\}$ to obtain $D((Q_n + e)/w, x)$, then we have

$$\begin{aligned} D((Q_n + e)/w, x) &= D(Q_n, x) + \frac{x}{x-1} [D(Q_{n-1}^\Delta, x) + D(Q_{n-1}^\Delta, x) - (Q_{n-1}^\Delta, x) \\ &\quad - D(Q_{n-1}^\Delta, x) - D(Q'_{n-2}, x) - D(Q'_{n-2}, x) + xD(Q'_{n-2}, x) + xD(Q'_{n-2}, x)] \\ &= D(Q_n, x) + 2xD(Q'_{n-2}, x). \end{aligned}$$

Therefore the result follows. \square

Theorem 7. *The domination polynomial of para-chain Q_n is given by*

$$\begin{aligned} D(Q_n, x) &= (x^3 + 2x^2 + x)D(Q_{n-1}, x) + (x^3 + 2x^2)D(Q_{n-2}, x) \\ &\quad + (x^3 + 3x^2)D(Q'_{n-2}, x) + (2x^4 + 4x^3)D(Q'_{n-3}, x), \end{aligned}$$

with initial conditions $D(Q_1, x) = x^4 + 4x^3 + 6x^2$ and $D(Q_2, x) = x^7 + 7x^6 + 21x^5 + 29x^4 + 15x^3$.

Proof. Consider the labeled Q_n as shown in Figure 4. We use Theorems 1 and 2 for vertex u_n to obtain the domination polynomial of Q_n . We have

$$\begin{aligned} D(Q_n, x) &= xD(Q_{n-1}^\Delta, x) + D(Q_{n-1}(2), x) + x^2D(Q'_{n-2}, x) - (1+x)xD(Q'_{n-2}, x) \\ &= xD(Q_{n-1}^\Delta, x) + D(Q_{n-1}(2), x) - xD(Q'_{n-2}, x). \end{aligned} \tag{2}$$

Therefore by parts (i), (ii) and (iv) of Lemma 1 and equation (2) we have

$$\begin{aligned} D(Q_n, x) &= x((1+x)D(Q_{n-1} + e, x) + xD(Q'_{n-2}, x)) + x(D(Q_{n-1} + e, x) \\ &\quad + D(Q_{n-1}, x) + D(Q'_{n-2}, x)) - xD(Q'_{n-2}, x) \\ &= (x^2 + 2x)D(Q_{n-1} + e, x) + x^2D(Q'_{n-2}, x) + xD(Q_{n-1}, x) \\ &= (x^2 + 2x)[x(D(Q_{n-1}, x) + D(Q_{n-2}, x)) + xD(Q'_{n-2}, x) \\ &\quad + 2x^2D(Q'_{n-3}, x)] + x^2D(Q'_{n-2}, x) + xD(Q_{n-1}, x) \\ &= (x^3 + 2x^2 + x)D(Q_{n-1}, x) + (x^3 + 2x^2)D(Q_{n-2}, x) \\ &\quad + (x^3 + 3x^2)D(Q'_{n-2}, x) + (2x^4 + 4x^3)D(Q'_{n-3}, x). \quad \square \end{aligned}$$

3.2 Domination polynomial of ortho-chain square cactus graphs

In this subsection we consider a ortho-chain of length n , O_n , as shown in Figure 7. We shall obtain a recurrence relation for the domination polynomial of O_n .

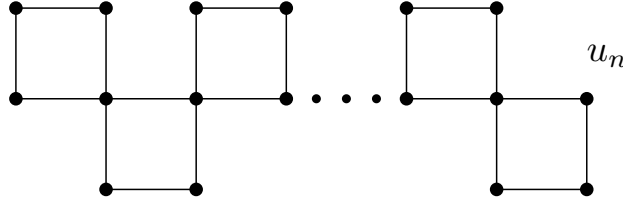


Figure 7: Labeled ortho-chain square O_n .

We need the following Lemma for finding domination polynomial of the O_n .

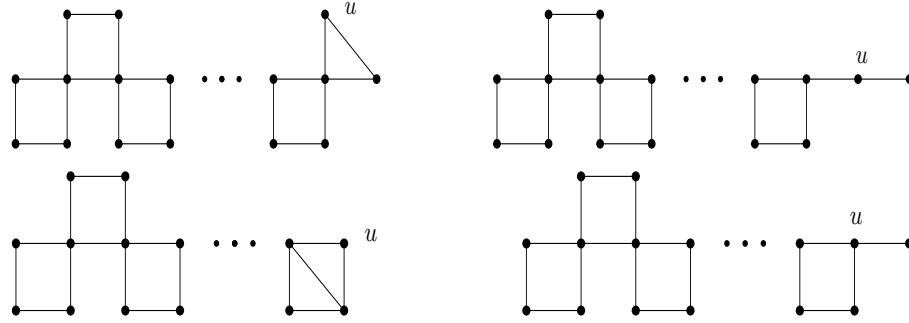


Figure 8: Graphs O_n^Δ , $O_n(2)$, O'_n and $O_n + e$, respectively.

Lemma 2. For graphs in figure 8 we have:

- (i) $D(O_n^\Delta, x) = (1+x)D(O_n + e, x) + xD(O_{n-1}(2), x)$, where $D(O_0^\Delta, x) = x^3 + 3x^2 + 3x$.
- (ii) $D(O_n(2), x) = x(D(O_n + e, x) + D(O_n, x) + D(O_{n-1}(2), x))$, where $D(O_0(2), x) = x^3 + 3x^2 + x$.
- (iii) $D(O'_n, x) = (1+x)D(O_n^\Delta, x) - xD(O_{n-1}(2), x)$, where $D(O'_0, x) = x^4 + 4x^3 + 6x^2 + 2x$.
- (iv) $D(O_n + e, x) = xD(O'_n, x) + xD(O_{n-1}(2), x) + x^2D(O_{n-2}(2), x)$, where $D(O_1 + e, x) = x^5 + 5x^4 + 9x^3 + 4x^2$.

Proof. The proof of parts (i), (ii) and (iv) follow from Theorems 1 and 2 for vertex u in graphs O_n^Δ , $O_n(2)$ and $O_n + e$, respectively. Note that in these cases $p_u(G, x) = 0$.

(iii) We use Theorems 1 and 2 for u in graphs O'_n . Since O'_n/u is isomorphic to $O'_n - u$ and $p_u(G, x) = xD(O_{n-1}(2), x)$. So we have the result. \square

Theorem 8. *The domination polynomial of para-chain O_n is given by*

$$D(O_n, x) = xD(O_{n-1}, x) + (x^2 + 2x)D(O_{n-1} + e, x) + x^2D(O_{n-2}(2), x),$$

with initial condition $D(O_1, x) = x^4 + 4x^3 + 6x^2$.

Proof. Consider the labeled O_n as shown in Figure 7. We use Theorems 1 and 2 for vertex u_n to obtain domination polynomial of O_n , then we have

$$\begin{aligned} D(O_n, x) &= xD(O_{n-1}^\Delta, x) + D(O_{n-1}(2), x) + x^2D(O_{n-2}(2), x) - (1+x)xD(O_{n-2}(2), x) \\ &= xD(O_{n-1}^\Delta, x) + D(O_{n-1}(2), x) - xD(O_{n-2}(2), x). \end{aligned}$$

Therefore by parts (i) and (ii) of Lemma 2 and this equation we have

$$\begin{aligned} D(O_n, x) &= x((1+x)D(O_{n-1} + e, x) + xD(O_{n-2}(2), x)) + x(D(O_{n-1} + e, x) \\ &\quad + D(O_{n-1}, x) + D(O_{n-2}(2), x)) - xD(O_{n-2}(2), x) \\ &= (x^2 + 2x)D(O_{n-1} + e, x) + x^2D(O_{n-2}(2), x) + xD(O_{n-1}, x). \quad \square \end{aligned}$$

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